

# Nonlinear Quantum Optical Spring

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## Abstract

The original idea of quantum optical spring arises from the requirement of quantization of the frequency of oscillations in Hamiltonian of harmonic oscillator. This purpose is achieved by considering a spring whose constant (and so its frequency) depends on the quantum states of another system. Recently, it is realized that by the assumption of frequency modulation of  $\omega$  to  $\omega\sqrt{1 + \mu a^\dagger a}$  the mentioned idea can be established. In the present paper we generalize the approach of quantum optical spring (has been called by us as nonlinear quantum optical spring) with attention to the *dependence of frequency to the intensity of radiation field* that *naturally* observed in nonlinear coherent states. Then, after the introduction of the generalized Hamiltonian of nonlinear quantum optical spring and its solution, we will investigate the nonclassical properties of the obtained states. Specially, typical collapse and revival in the distribution functions and squeezing parameters as particular quantum features will be revealed.

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**Keywords:** Nonlinear coherent states, Quantum optical spring, Nonclassical states.

## 1 Introduction

Quantization of harmonic oscillator's Hamiltonian with the definitions of creation and annihilation bosonic operators is achieved. The spectrum of the system is discrete as  $(n + \frac{1}{2})\hbar\omega$ . An important point is that in quantum and classical Hamiltonians the frequency of oscillations ( $\omega = \sqrt{k/m}$ ) is not quantized. The main idea of quantum optical spring lies in the quantization of frequency of oscillations. Recently Rai and Agarwal have designed a quantum optical spring such that spring constant depends on the quantum state of another system in a special form [1]. They realized this phenomenon by replacing  $\omega$  with

$\omega\sqrt{1+\mu n}$ , where  $n = a^\dagger a$  is the number operator. Then, the Hamiltonian of quantum optical spring is introduced as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(1+\mu n)x^2. \quad (1)$$

A factor which generalizes spring constant has been called quantized source of modulation (QSM); in this case  $\mu n$ . Therefore, the eigenstate of the whole system is obtained by multiplying the number state with the eigenfunctions of harmonic oscillator [1].

## 2 Introducing nonlinear quantum optical spring

It is clear that the main work of Rai and Agarwal [1] can be summarized in imposing the intensity dependence of frequency of a quantum harmonic oscillator. Hence, in a sense we may call this further quantization as a second quantization type. With appropriate physical motivations of the transformation has been done in Hamiltonian of quantum optical spring according to relation (1) the authors obtained some elegant and interesting results. One can extend the special transformation from  $\omega^2$  to  $\omega^2(1+\mu n)$ , to a generalized transformation  $\omega^2(1+F(n))$ , where  $F(n)$  is an appropriate function of number operator.

Now, recall that in the core of the nonlinear coherent states in quantum optics there exist an operator-valued function  $f(n)$  responsible for the nonlinearity of the oscillator algebras [2, 3]. These states attracted much attention in recent decades [4, 5]. As one of the special features of these states we may refer to intensity dependence of the frequency of nonclassical lights. This important aspect of nonlinear coherent states has been apparently clarified for the  $q$ -deformed coherent states with the particular nonlinearity function [6]. The main goal of our presentation is to establish a natural link between "quantum optical spring" and "nonlinear coherent states" associated to nonlinear oscillator algebra, the idea that naturally leads one to a general formalism for "*nonlinear quantum optical spring*". It is worth to mention that although we emphasis on the nonlinearity of the quantum optical spring, as we will establish in the continuation of the paper, the special case introduced by Rai and Agarwal is also nonlinear with a special nonlinearity function. Accordingly, our work is not a generalization from linear to nonlinear quantum optical spring. Instead, along finding the natural link between the quantum optical spring and nonlinear CSs, so in addition to enriching the physical basis of such systems a variety of quantum optical springs may be constructed.

Single-mode nonlinear coherent states are known with deformed ladder operators  $A = af(n)$  and  $A^\dagger = f(n)a^\dagger$  where  $f(n)$  is an operator-valued function responsible for the nonlinearity of the system. We assumed  $f(n)$  to be real. A suitable description for the dynamics of the nonlinear oscillator is

$$H = \omega A^\dagger A. \quad (2)$$

Before paying attention to the main goal of the present paper it is reasonable to have a brief discussion on the form of the Hamiltonian in the nonlinear coherent states approach. The Hamiltonian  $H_M = \frac{\omega}{2}(A^\dagger A + AA^\dagger)$  has been introduced in [2] in analogous to the quantized harmonic oscillator formalism. We have previously established in [7] that requiring the "action identity" criterion on the nonlinear coherent states leads us to the simple form of it as expressed in (2). This proposal is consistent with the ladder operators formalism and Hamiltonian definition have been outlined in *super-symmetric quantum mechanics* contexts in the literature [8, 9]. It is worth also to notice that recently a general formalism for the construction of coherent state as eigenstate of the annihilation operator of the "generalized Heisenberg algebra" (GHA) is introduced [10]. There are some physical examples there. It is easy to investigate that the "nonlinear coherent states" for single-mode nonlinear oscillators, as the algebraic generalization of standard coherent states, may be consistently placed in GHA structure only if one takes the Hamiltonian associated to nonlinear oscillators as in (2). Indeed, one must consider  $\{A, A^\dagger, J_0\}$  with  $J_0 = H$  as the generators of the GHA. After all, our proposal allows us to relate simply the nonlinear coherent states to the one-dimensional solvable quantum systems with known discrete spectrum, i.e.,  $f(n) = \sqrt{e_n/n}$ , where  $H|n\rangle = e_n|n\rangle$ .

Anyway, the time evolution operator

$$U(t) = \exp(-iH(n)t/\hbar) \quad (3)$$

gives the following expression for the time evolved operator  $A(t)$

$$A(t) = U^\dagger(t)AU(t) = A \exp(-i\omega\Omega(n)t) \quad (4)$$

where

$$\Omega(n) \equiv (n+1)f^2(n+1) - nf^2(n). \quad (5)$$

The latter relation indicates that frequency of oscillations depends explicitly on the intensity. Now we return to the Hamiltonian of harmonic oscillator and deform it as follows

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\Omega^2(n)x^2 \quad (6)$$

where the term  $\Omega^2(n) - 1$  in the above Hamiltonian plays the role of QSM. The Hamiltonian expression in (6) describes the dynamics of the nonlinear quantum optical spring. Using the known solutions of harmonic oscillator with eigenfunction  $\phi_n(x)$  we obtain the solutions of Schrödinger equation for modulated Hamiltonian as  $\mathcal{H}\psi_n^p|p\rangle = E_n^{(p)}\psi_n^p|p\rangle$  as

$$\begin{aligned} \psi_n^p &= N_n H_n(\alpha_p x) \sqrt{\frac{\alpha_p}{\alpha}} \exp\left(-\frac{1}{2}\alpha_p^2 x^2\right) \\ E_n^p &= \hbar\omega\Omega(p)\left(n + \frac{1}{2}\right) \end{aligned} \quad (7)$$

where  $H_n$  is the  $n$ th order of Hermite polynomials,  $a^\dagger a |p\rangle = p |p\rangle$ ,  $\alpha_p \equiv (\frac{m\omega\Omega(p)}{\hbar})^{\frac{1}{2}}$ ,  $N_n = (\frac{\alpha}{\sqrt{\pi}2^n n!})^{\frac{1}{2}}$  and  $\Omega(p)$  introduced in (5). Note that  $\psi_n^p$  is the eigenstate of harmonic oscillator with frequency  $\omega$  replaced by  $\omega\Omega(p)$ , and the energy eigenvalues of the modulated Hamiltonian in (7) characterized by two quantum numbers  $n$  and  $p$ . For a fixed  $p$  these states form a complete set. Obviously, from Eq. (5) it may be seen that if  $f(n) = 1$  then  $\Omega(n) = 1$  and so  $\psi_n^p$  simplifies to  $\phi_n(x)$ . From now on we will follow nearly similar procedure of Rai and Agarwal with the same initial state for the generalized modulated system we introduced in (1) as

$$\psi(t=0) = \sum_{p,n} C_{pn} \phi_n(x) |p\rangle. \quad (8)$$

Making use of the time evolution operator with the Hamiltonian in (6) on Eq. (8) we obtain

$$|\psi(t)\rangle = \sum_{p,n,l} C_{pn} \exp\left(\frac{-iE_l^p t}{\hbar}\right) \langle\psi_l^p|\phi_n\rangle |p\rangle |\psi_l^p\rangle. \quad (9)$$

For next purposes it is required to derive the density matrix with the following result

$$\rho_0 = \sum_{n,l,m,j,p} |\psi_l^p\rangle \langle\psi_j^p| C_{pn} C_{pm}^* \exp\left[\frac{-i(E_l^p - E_j^p)t}{\hbar}\right] \langle\psi_l^p|\phi_n\rangle \langle\phi_n|\psi_j^p\rangle. \quad (10)$$

Eq. (10) helps us to study the quantum dynamics of the oscillator coupled to QSM. It is easy to check that setting

$$f_{RA}(n) = \left(\frac{\sum_{j=0}^{n-1} \sqrt{1+\mu j}}{n}\right)^{\frac{1}{2}} = \left[\frac{\sqrt{\mu} \left(\zeta(-\frac{1}{2}, \frac{1}{\mu}) - \zeta(-\frac{1}{2}, n + \frac{1}{\mu})\right)}{n}\right]^{\frac{1}{2}} \quad (11)$$

or equivalently  $\Omega_{RA}(n) = \sqrt{1+\mu n}$  in all above relations leads to the recent results of Rai and Agarwal in [1], where  $\zeta(m, n)$  is the well-known Zeta function. We would like to end this section with mentioning that choosing different  $f(n)$ 's leads to distinct nonlinear quantum optical springs. So our proposal can be actually considered as the generalization of their work.

### 3 Quantum dynamics of the nonlinear quantum optical spring

Let us consider the situation where QSM and the oscillator are respectively prepared in a coherent state and in its ground state. Thus, one has

$$C_{pn} = \delta_{n0} \frac{\alpha^p \exp(-\frac{|\alpha|^2}{2})}{\sqrt{p!}}. \quad (12)$$

It must be noted the equations (12) and (8) determine the initial states. So no relation between these terms and the evolution Hamiltonian described the nonlinear oscillator may be expected. Inserting (12) into Eq. (10) we get

$$\rho_0 = \sum \frac{|\alpha|^{2p} \exp(-|\alpha|^2)}{p!} \exp[-i\omega\Omega(p)t(l-j)] |\psi_l^p\rangle \langle\psi_j^p| \langle\psi_l^p|\phi_0\rangle \langle\phi_0|\psi_j^p\rangle. \quad (13)$$

The probability of finding the oscillator in the initial state is obtained by

$$P_0(t) = \langle\phi_0|\rho_0|\phi_0\rangle = \sum_p |A_p|^2 Q(p) \quad (14)$$

where  $Q(p) = \frac{|\alpha|^{2p} \exp(-|\alpha|^2)}{p!}$  is the Poissonian distribution function and

$$A_p = \sum_l \exp[-i\omega\Omega(p)tl] |\langle\psi_l^p|\phi_0\rangle|^2. \quad (15)$$

The latter formula is one of our key results for the description of nonclassical properties of the nonlinear quantum optical spring. Now by calculating  $\langle\psi_l^p|\phi_0\rangle$  in (15) one finally arrives at

$$A_p = \frac{|\beta_p|^2}{\Omega(p)^{\frac{1}{2}} [1 - (\beta_p^2 - 1)^2 \exp(-2i\omega\Omega(p)t)]^{\frac{1}{2}}} \quad (16)$$

where we set

$$\beta_p^2 \equiv \frac{2\Omega(p)}{1 + \Omega(p)}. \quad (17)$$

For classical source of modulation it is enough to replace  $p$  by  $|\alpha|^2$  in Eq. (14). Consequently

$$P_{cl}(t) = \frac{|\beta_\alpha|^2}{\sqrt{\Omega^2(|\alpha|^2)(1 - 2(\beta_\alpha^2 - 1)^2 \cos(2\omega_\alpha t) + (\beta_\alpha^2 - 1)^4)}} \quad (18)$$

where  $\omega_\alpha \equiv \omega\Omega(|\alpha|^2)$ . Clearly  $P_{cl}$  oscillates at frequency  $2\omega_\alpha$ .

## 4 Squeezing properties of nonlinear quantum optical spring

Since the number of photons and so the Hamiltonian operator in (2) is constant, we have the following relations

$$\begin{aligned} x(t) &= x(0) \cos(\omega\Omega(n)t) + \frac{p(0)}{m\omega\Omega(n)} \sin(\omega\Omega(n)t) \\ p(t) &= p(0) \cos(\omega\Omega(n)t) - m\omega\Omega(n)x(0) \sin(\omega\Omega(n)t). \end{aligned} \quad (19)$$

Now we define the squeezing parameters as

$$S_x(t) = \frac{\langle x^2(t) \rangle - \langle x(t) \rangle^2}{\langle x^2(0) \rangle}, \quad S_p(t) = \frac{\langle p^2(t) \rangle - \langle p(t) \rangle^2}{\langle p^2(0) \rangle}. \quad (20)$$

The expectation values must be calculated with respect to the states in (8). As a result it is easy to show

$$\begin{aligned} S_x(t) &= 1 - \sum_n \frac{|\alpha|^{2n} \exp(-|\alpha|^2) \sin^2(\omega \Omega(n)t)}{n!} \left(1 - \frac{1}{\Omega^2(n)}\right) \\ S_p(t) &= 1 + \sum_n \frac{|\alpha|^{2n} \exp(-|\alpha|^2) \sin^2(\omega \Omega(n)t)}{n!} (\Omega^2(n) - 1). \end{aligned} \quad (21)$$

Note that unlike the special case considered by Rai and Agarwal with  $\Omega^2(n) = 1 + \mu n$  for which  $S_x$  is always less than one ( $x$ -quadrature is always squeezed) and hence  $p$ -quadrature is not squeezed at all, that is not so in general. All we may conclude from the two latter relations are that squeezing in both quadratures may be occurred (certainly not simultaneously) depending on the selected nonlinearity function  $f(n)$ .

## 5 Physical applications of the formalism

Now we like to apply the presented formalism to a few classes of nonlinear coherent states. There exist various nonlinearity functions in the literature which have been introduced for different purposes, mainly due to their nonclassical properties. Among them we only deal with two classes of them, i.e., " $q$ -deformed coherent states" and "photon-added coherent states".

- *$q$ -deformed coherent states*

As the first example we use the  $q$ -deformed nonlinearity function. Recall that Man'ko *et al* showed that  $q$ -coherent states are indeed nonlinear coherent states with nonlinearity function [6]

$$f_q(p) = \sqrt{\frac{q^p - q^{-p}}{p(q - q^{-1})}} = \sqrt{\frac{\sinh(\lambda p)}{p \sinh \lambda}} \quad (22)$$

where  $q = \ln \lambda$ . The corresponding states have frequency dependence on intensity of radiation field (blue shift) [6]. By replacing Eq. (22) into (5) one readily gets

$$\Omega_q(p) = \frac{\cosh \left[ \frac{\lambda}{2} (2p + 1) \right]}{\cosh \frac{\lambda}{2}} \quad (23)$$

- *Photon added coherent states*

As the next example we will consider photon-added coherent states (PACSs) first introduced by Agarwal and Tara [11] as  $|\alpha, m\rangle = N(a^\dagger)^m|\alpha\rangle$ , where  $|\alpha\rangle$  is the canonical coherent states,  $N$  is an appropriate normalization factor and  $m$  is a non-negative integer. Sivakumar realized that these states satisfy the eigen-value equation  $A|\alpha, m\rangle = \alpha|\alpha, m\rangle$  where  $A = f(n, m)a$  [4]. Hence, these states are non-linear coherent states with nonlinearity function

$$f_{PACS}(n, m) = 1 - \frac{m}{1+n}. \quad (24)$$

Thus, by replacing (24) into Eq. (5) we have

$$\Omega_{PACS}(p, m) = (p+1) \left( \frac{p+2-m}{p+2} \right)^2 - p \left( \frac{p+1-m}{p+1} \right)^2. \quad (25)$$

Also, it is shown that the operator-valued function associated with  $|\alpha, -m\rangle$  is  $f_{PACS}(n, -m) = 1 + m/(1+n)$ . Replacing  $m$  by  $-m$  in (25) one may obtain  $\Omega_{PACS}(p, -m)$  associated to  $|\alpha, -m\rangle$  states [4].

Our numerical results have been displayed in figures 1 and 2 show respectively the probability distribution function and squeezing parameter in  $x$ -quadrature against  $\tau$  for  $q$ -deformed coherent states with  $\Omega_q(p)$  introduced in (23) and the choosed parameters (note that the mean photon number is indeed  $|\alpha|^2$  and  $\tau = \omega t/(2\pi)$  in all figures). A typical collapse and revival exhibition may be observed from the two figures, which is due to the discrete nature of the quantum state of the source of modulation (quantization of the frequency of the oscillation). In figures 3-a (for the states  $|\alpha, m\rangle$ ) and 3-b, 3-c (for the states  $|\alpha, -m\rangle$ ) we displayed the probability distribution as a function of  $\tau$  for the different parameters. Also in figure 4 (for the states  $|\alpha, m\rangle$ ), and figures 5-a, 5-b, 5-c (for the states  $|\alpha, -m\rangle$ ) the squeezing parameter in  $x$ -quadrature have been shown against  $\tau$  for the choosed parameters. A typical collapse and revival exhibition can be observed from all figures, which again have their roots in the above mentioned source, i.e., the spring constant is controlled by another quantum source (QSM). From figures 4, 5-b (for the states  $|\alpha, -m\rangle$ ) the squeezing exhibition may be observed, while this nonclassical feature may not be seen from figures 5-a and 5-c. It is worth to notice that although the two latter figures do not show squeezing feature, the collapses and revivals as quantum features of the special type of nonlinear quantum optical spring are visible. To this end we would like to mention that as we stated before all cases considered by us and the special case of Rai and Agarwal quantum optical springs are nonlinear in nature according to our terminology, so the general features of all are the same at least qualitatively.

## 6 Conclusion

The presented formalism in the present manuscript compare to Rai and Agarwal method has the advantage that the quantization of frequency of oscillations which comes out naturally from the nonlinear coherent states approach were imposed on the quantized Hamiltonian of harmonic oscillator. So in this way we impose a second quantization type on the quantized harmonic oscillator. Also our approach can be easily used for a wide range of nonlinear oscillators as well as every solvable quantum systems, due to the simple relation  $e_n = n f^2(n)$  [12, 13, 9]. This approach leads to typical collapses and revivals in probability distribution function  $P_0$ . The latter results in addition to the squeezing parameter  $S_x$  indicate that these are purely quantum mechanical phenomena.

## Acknowledgements

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## FIGURE CAPTIONS:

**FIG. 1** The variation of  $P_0$  as a function of  $\tau$  with parameters  $\lambda = 0.1$  and  $\alpha = 1.25$  for QSM of  $q$ -deformed coherent states.

**FIG. 2** The variation of  $S_x(t)$  as a function of  $\tau$  with parameters  $\lambda = 0.1$  and  $\alpha = 1.25$  for QSM of  $q$ -deformed coherent states.

**FIG. 3-a** The variation of  $P_0$  as a function of  $\tau$  with parameters  $m = 1$  and  $\alpha = 3$  for QSM of  $|\alpha, m\rangle$  states.

**FIG. 3-b** The variation of  $P_0$  as a function of  $\tau$  with parameters  $m = 1$  and  $\alpha = 3$  for QSM of  $|\alpha, -m\rangle$  states.

**FIG. 3-c** The variation of  $P_0$  as a function of  $\tau$  with parameters  $m = 4$  and  $\alpha = 3$  for QSM of  $|\alpha, -m\rangle$  states.

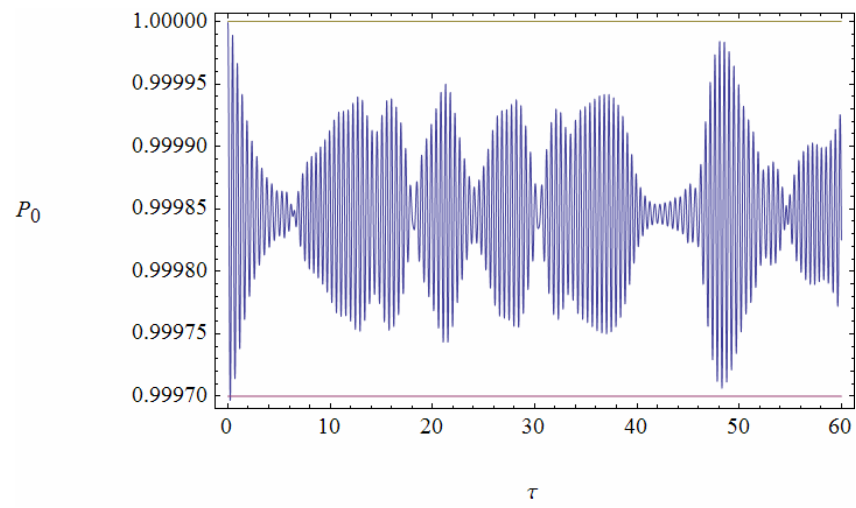
**FIG. 4** The variation of  $S_x(t)$  as a function of  $\tau$  with parameters  $m = 1$  and  $\alpha = 2$  for QSM of  $|\alpha, -m\rangle$  states.

**FIG. 5-a** The variation of  $S_x(t)$  as a function of  $\tau$  with parameters  $m = 1$  and  $\alpha = 3$  for QSM of  $|\alpha, m\rangle$  states.

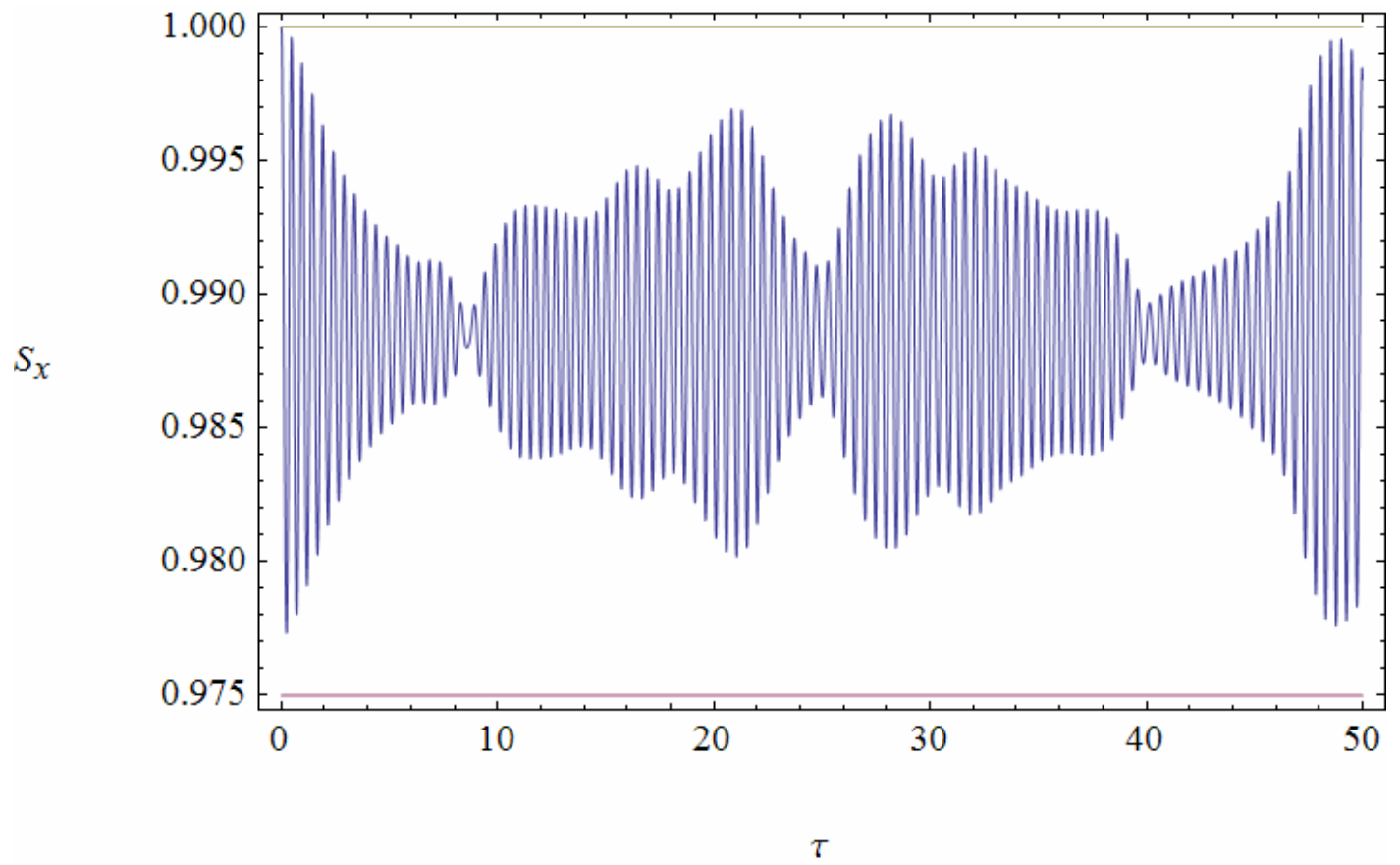
**FIG. 5-b** The variation of  $S_x(t)$  as a function of  $\tau$  with parameters  $m = 1$  and  $\alpha = 3$  for QSM of  $|\alpha, -m\rangle$  states.

**FIG. 5-c** The variation of  $S_x(t)$  as a function of  $\tau$  with parameters  $m = 4$  and  $\alpha = 3$  for QSM of  $|\alpha, -m\rangle$  states.

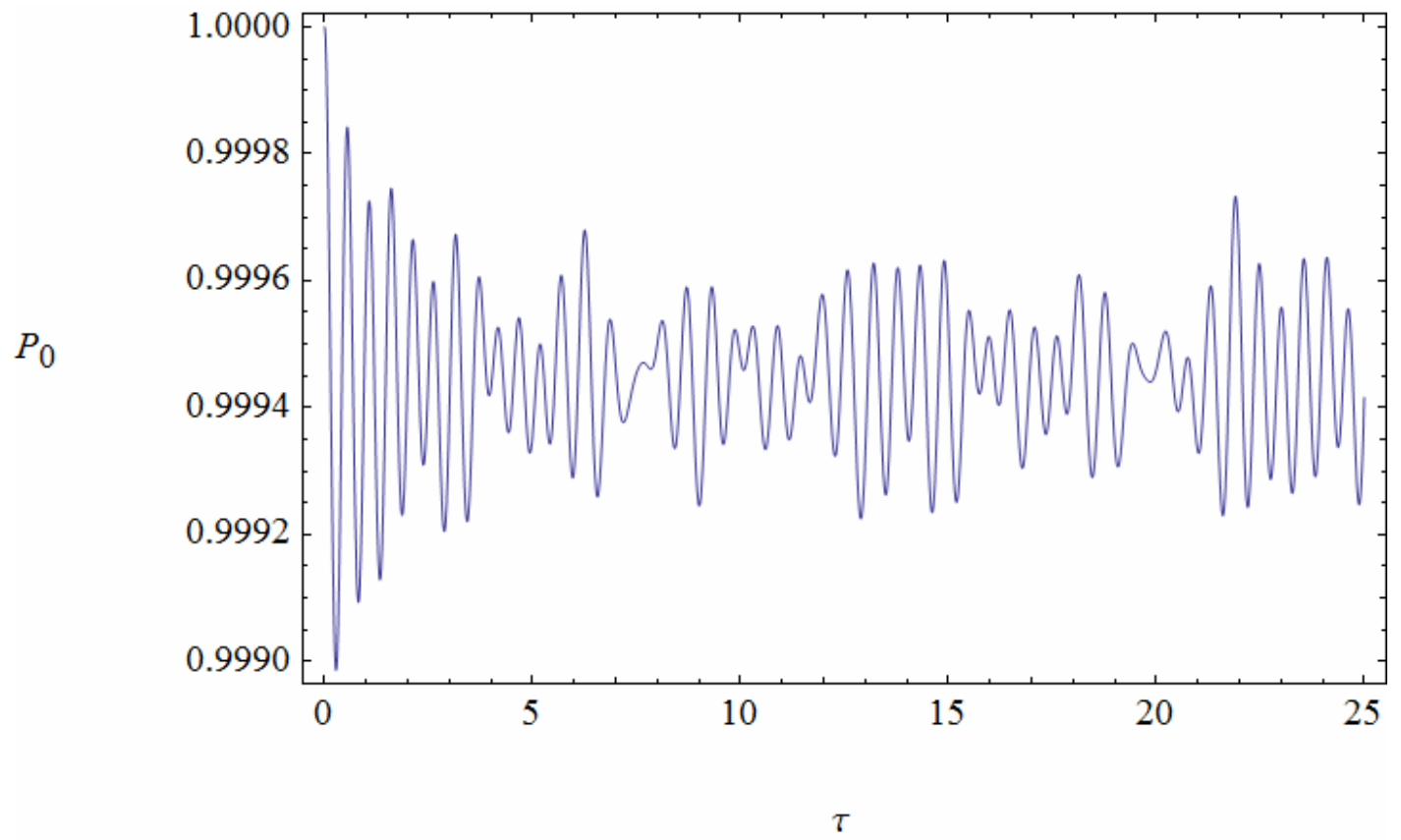
**Figure 1**



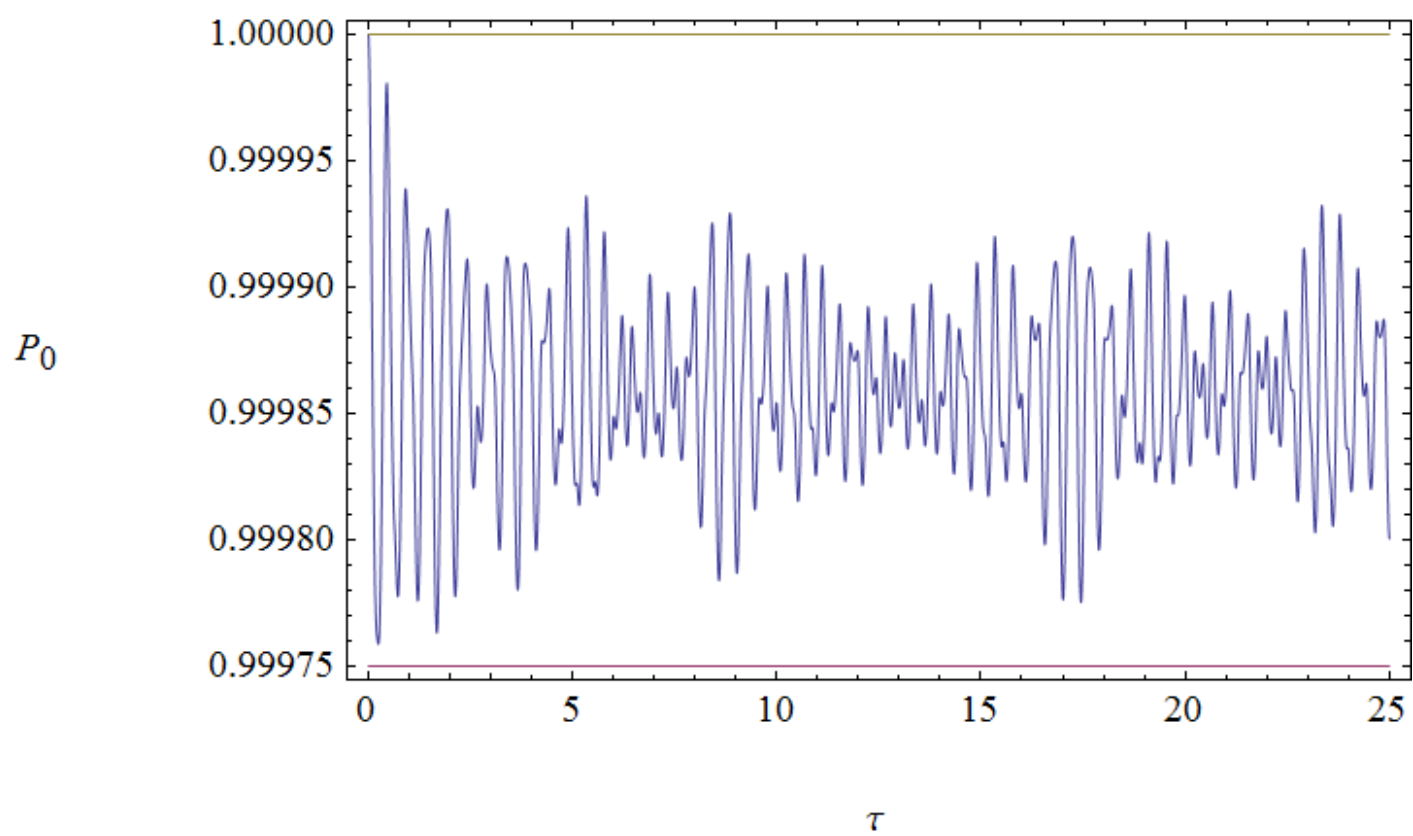
**Figure 2**



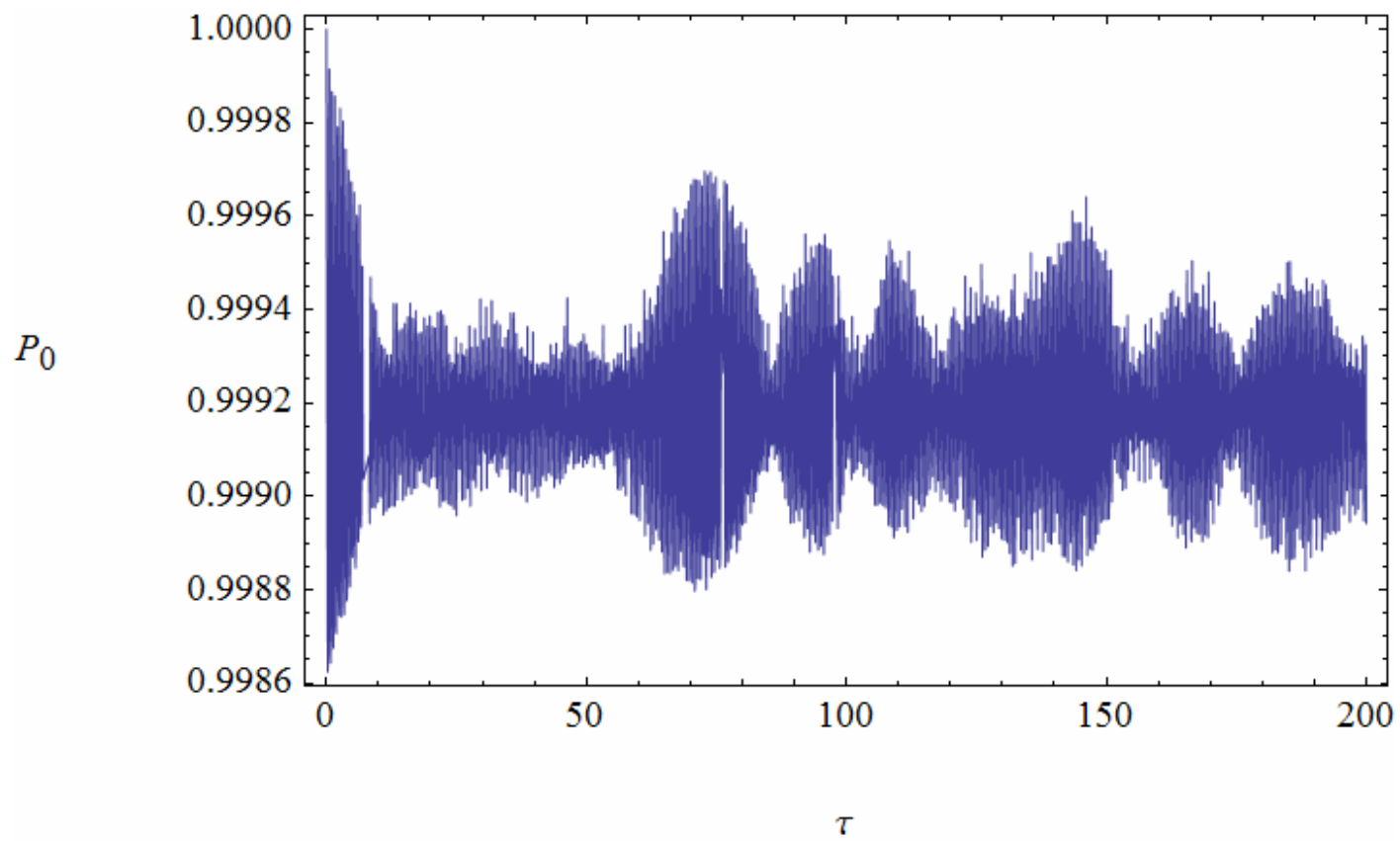
**Figure 3-a**



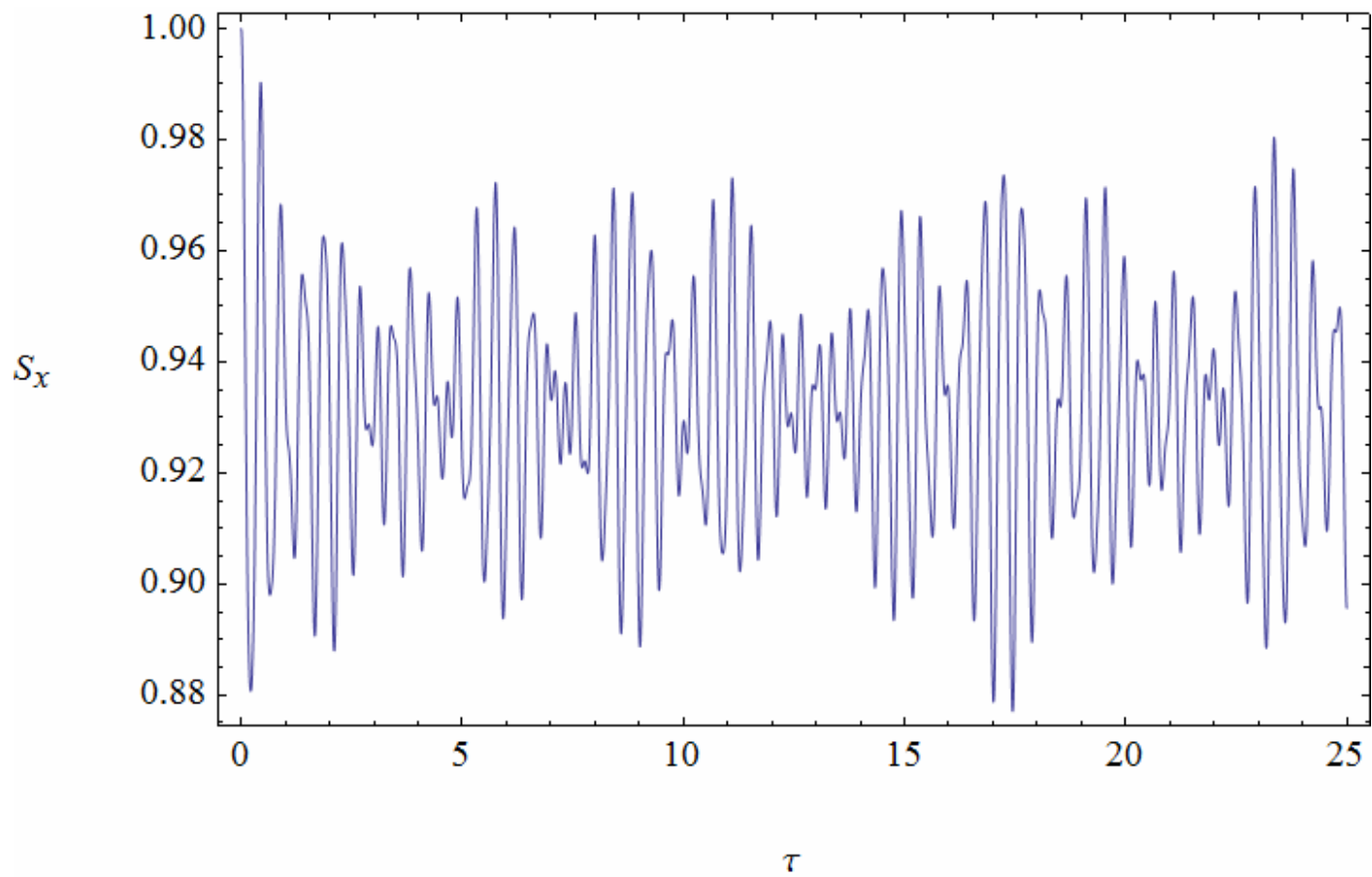
**Figure 3-b**



**Figure 3-c**

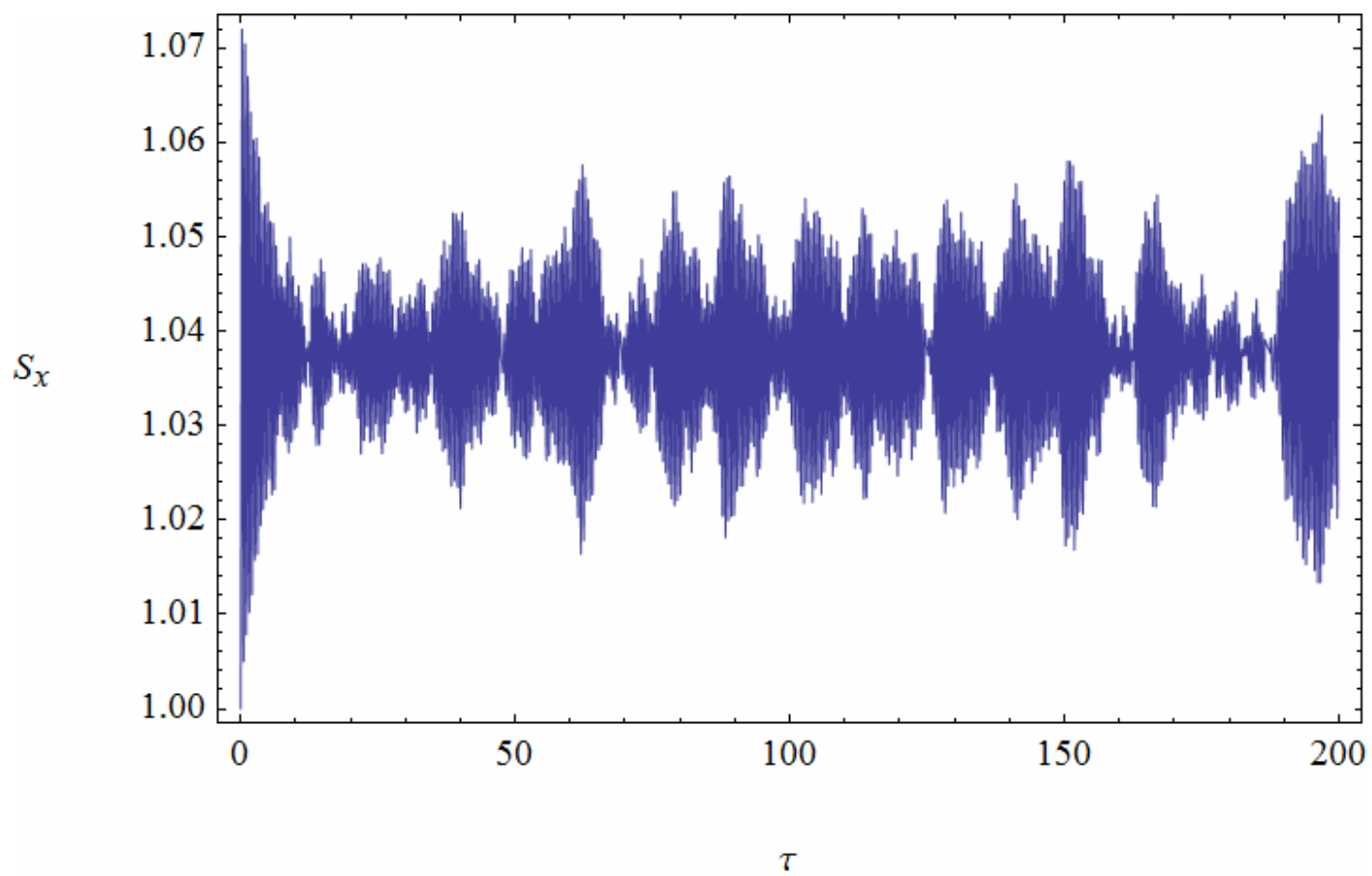


**Figure 4**

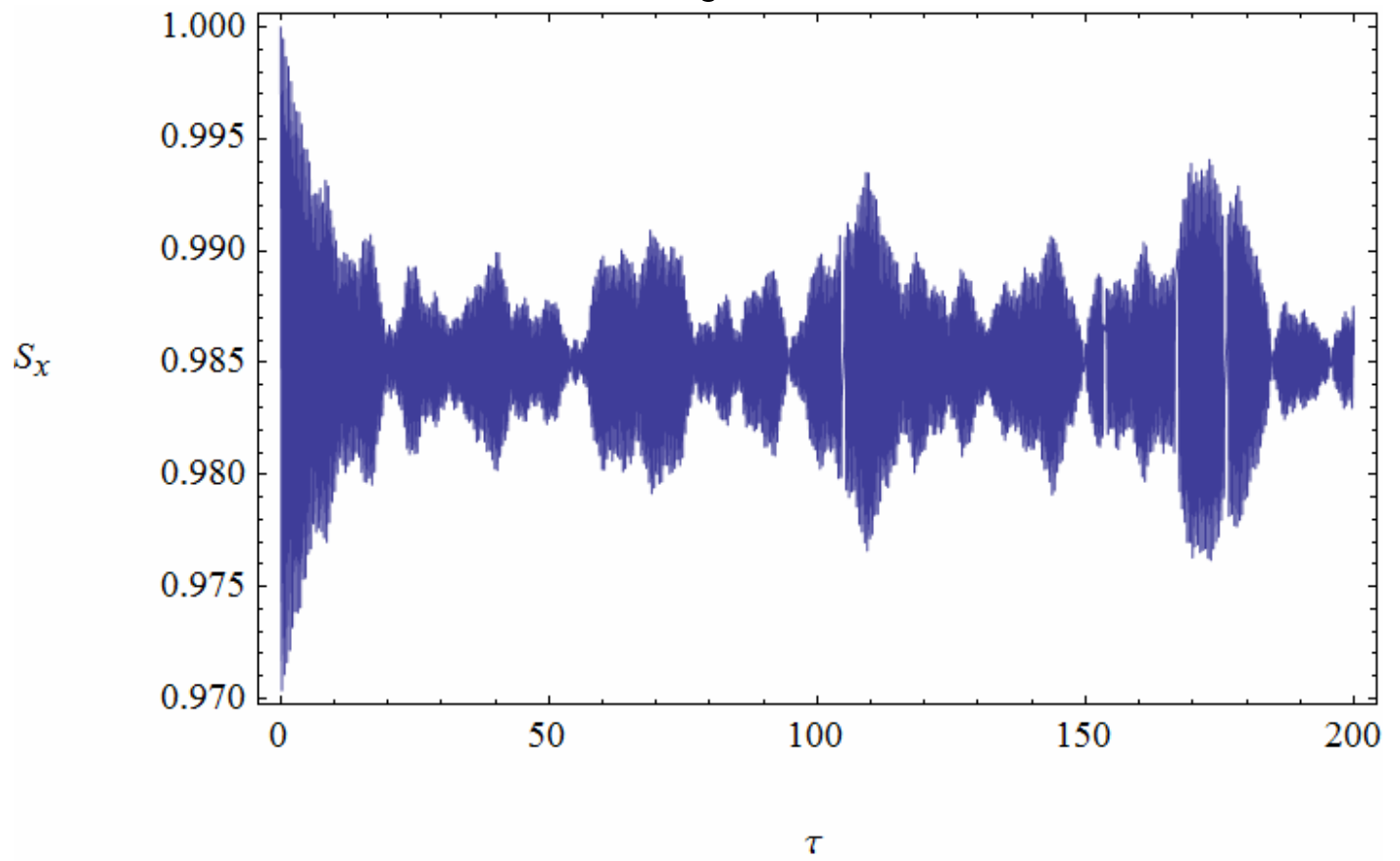




**Figure 5-a**



**Figure 5-b**



**Figure 5-c**

